Binary black hole shadows, the Cantor set and the Lakes of Wada

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Talk Outline

1. Introduction to black hole shadows
2. Toy Model: Majumdar-Papapetrou
3. Rays in the plane: symbolic dynamics and the Cantor set
4. Non-planar rays and 2D shadows
5. Stable bounded null orbits
6. The Lakes of Wada
7. On (non-)existence of stable null orbits
8. Conclusions
Why study binary black hole shadows?

Binary BHs are interesting
- GW150914: BHs exist in binary pairs.
- BH mergers \(\Rightarrow\) Gravitational waves

BH shadows will be observed
- Image of BH at centre of galaxy.
- We ‘heard’ BHs at LIGO.
  Will we see BHs with the EHT?

Binary BH shadows are fascinating
- Naturally-occurring fractals.
- Exemplar of chaotic scattering.

Event Horizon Telescope.  
[Credit: ESO/L. Calçada]
Introduction to black hole shadows

What is a BH shadow?
- Ray-casting: null rays traced away from camera lens, backwards in time.
- **Shadow** = region of initial data where rays are traced back to BH horizon.

What does a binary shadow look like?
- **Not** just superposition of 2 singleton shadows.
- **Eyebrow-like** features.
- **Self-similar** structure.
- Binary shadow is **fractal**.

[A. Bohn, W. Throwe, F. Hébert *et al.*, *Class. Quant. Grav.* **32** 065002 (2015)]

Binary shadow with eyebrows.
[Credit: SXS Lensing Group, www.black-holes.org]
Singleton BH shadows

 Integrable geodesic equations $\iff$ regular structure $\iff$ boring!
Double BH shadows

Non-integrable system ⇔ chaotic scattering ⇔ fractal structure
2. The Majumdar–Papapetrou di-hole
Toy model

- By playing with a toy model we may understand qualitative features.
- I will use the **Majumdar-Papapetrou di-hole** solution.

Fate of null geodesics:

1. fall into BH1
2. fall into BH2
3. escape to \( \infty \)

- Unstable **perpetual orbits** – neither scattered nor absorbed.

Toy model

- **Majumdar–Papapetrou spacetimes** are static asymptotically-flat extremally-charged solutions in electrovacuum:

\[
\begin{align*}
    ds^2 &= -U^{-2}dt^2 + U^2 d\mathbf{x} \cdot d\mathbf{x}, \\
    A_\mu &= [U^{-1}, 0].
\end{align*}
\]

- \( U(\mathbf{x}) \) is **any** solution of Laplace’s equation: \( \nabla^2 U = 0 \).

- **Linearity** ⇔ superposition ⇔ gravitational attraction in balance with electrostatic repulsion.

- Null geodesics are governed by a Maupertuis’ principle:

\[
    S = \int n(\mathbf{x}) dl.
\]

- Spacetime has an **effective refractive index** \( n(\mathbf{x}) = U^2(\mathbf{x}) \).
Toy model

- The **Majumdar–Papapetrou di-hole** solution

\[ U(x) = 1 + \frac{M_-}{\sqrt{\rho^2 + (z - z_-)^2}} + \frac{M_+}{\sqrt{\rho^2 + (z - z_+)^2}}. \]

where

\[ z_\pm = \frac{\pm a M_\mp}{M_+ + M_-} \]

- Two black holes of mass \( M_\pm = Q_\pm \) separated by coordinate distance \( a \), with centre of mass at the origin.

- The black hole horizons (**not** singularities) are at \( \rho = 0, \ z = z_\pm \).

- We focus on equal-mass case \( M_\pm = 1 \).
3. Rays in the plane
Null geodesics in a plane

A one-dimensional example

- **Left**: Rays starting at the centre of mass with opening angle $\alpha$
- **Right**: The fate of the rays as a function of $\alpha$. 
Symbolic dynamics (I)

Make a decision . . .

0 Go around other BH, same sense.
1 Fall into a BH (absorbed).
2 Go around other BH, opposite sense.
3 Escape to $\infty$ (scattered).
4 Go around same BH again.
Symbolic dynamics (II)

Describing null geodesics
- Encode rays as base-5 sequences.
- Rays ending in 1 are in the shadow.
- Rays ending in 3 escape to \( \infty \) (not in the shadow).
- **Perpetual orbits**: infinitely-long sequences **without** digit 1 or 3.

Any ray in the plane can be described as a sequence of decisions, e.g. 243 or 20400421 or 00020444240 \( \cdots \).
Symbolic dynamics (III)

‘Decision dynamics’

- We call this decision dynamics
- An example of symbolic dynamics
- A topological description
- Other versions available (e.g. Cornish & Gibbons)
- Our sequences correspond to† the ordering in initial data
There exists...

- A countably-infinite set of **periodic** orbits (cf. rational numbers)
- A uncountably-infinite set of **aperiodic** orbits (cf. irrational numbers)
Cantor set construction (I)
Cantor set construction (II)
Shadow construction (I)
The shadow is the union of open intervals ending in the digit 1.
Magnify a coloured region to see self-similarity.

- e.g. zoom in on the right-hand (green) region (decision 4).
Chaotic scattering (Eckhardt 1988)

- Scattering in a Hamiltonian system is *irregular* (or chaotic) if there exists, on some manifold of initial data, an infinity of distinct ‘scattering singularities’ of measure zero, typically arranged into a fractal set.

- A ‘scattering singularity’ is an initial value for which the scattering process is not defined, and some physical quantity such as deflection angle or time delay becomes singular.
Chaotic scattering and binary BH shadows

- Static binary BHs exhibit chaotic scattering.
- Each distinct perpetual orbit generates one ‘scattering singularity’ in an initial data set.
- i.e., an initial value for a ray which asymptotes towards a perpetual orbit.
- Any open interval in $\alpha$ contains either zero or infinitely-many scattering singularities.
- The set of scattering singularities (cf. the future invariant set) is called a strange repellor.

The scattering singularities...

- ...have a Cantor-like distribution in the initial data.
- ...have zero (Lebesque) measure.
- ...are ordered in the same way as in our base-5 code†.

† modulo a parity-reversal operation, see paper for details.
4. Non-Planar Rays

Rays with angular momentum about the symmetry axis
Null geodesic equations

- A geodesic is a spacetime path $q^\mu(\lambda)$ that extremizes the action:

$$S[q^\mu(\lambda)] = \int L(q^\mu, \dot{q}^\mu) d\lambda, \quad \dot{q}^\mu \equiv \frac{dq^\mu}{d\lambda}$$

- Lagrangian: $L = \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu$
- Conjugate momenta: $p_\mu \equiv \frac{\partial L}{\partial \dot{q}^\mu}$
- Hamiltonian: $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$.
- Hamilton’s equations: $\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}, \dot{p}_\mu = -\frac{\partial H}{\partial q^\mu}$.

Symmetries

- **Null geodesics**: $\Rightarrow L = 0, H = 0$.
- **Static**: set $p_t = -1$ without loss of generality
- **Axisymmetric**: $p_\phi = \text{const.}$
A 2D non-integrable Hamiltonian

- Start with static axisymmetric spacetime in Weyl coordinates:
  \[ ds^2 = -U^{-2} dt^2 + U^2 \left( e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2 \right) \]
  (N.B. \( e^{2\gamma} = 1 \) in MP spacetime).
- Apply a conformal transformation, \( ds^2 = (Ue^{\gamma})^2 d\tilde{s}^2 \):
  \[ d\tilde{s}^2 = d\rho^2 + dz^2 + e^{-2\gamma} \left( -U^{-4} dt^2 + \rho^2 d\phi^2 \right) \]
- Get 2D Hamiltonian with one parameter \( p_{\phi} \) and null condition \( H = 0 \):
  \[ H = \frac{1}{2} (p_{\rho}^2 + p_z^2) + V(\rho, z), \quad V(\rho, z) = \frac{1}{2\rho^2} e^{2\gamma} (p_{\phi}^2 - \rho^2 U^4) \].
- Introducing a **height function** \( h(\rho, z) \equiv \rho U^2 \),
  \[ V = \frac{1}{2\rho^2} e^{2\gamma} (h + p_{\phi}) (h - p_{\phi}) \]
- The contours \( h(\rho, z) = p_{\phi} \) demarcate the regions in \( (\rho, z) \) forbidden by angular momentum.
Examples of non-planar periodic orbits (I)
Examples of non-planar periodic orbits (II)

222 \cdots
Examples of non-planar periodic orbits (III)
Examples of non-planar periodic orbits (IV)

0202 \cdots
Angular momentum $p_\phi$ and fundamental orbits (I)

\[ p_\phi = 4.0 \quad \text{and} \quad p_\phi = 5.0 \]
Angular momentum $p_{\phi}$ and fundamental orbits (II)

\[ p_{\phi} = 5.08 \]

\[ p_{\phi} = 5.9 \]
Two-dimensional shadows: $\alpha = 2$
2D shadow: $\theta = 0^\circ$, $a = 2$
2D shadow: $\theta = 90^\circ$, $a = 2$

- Each 1D slice is at constant $p_\phi$.
- The black and orange slices are Cantor-like.
- The pink slice is regular (non-fractal).
Separation of black holes $a$ and critical contours of $h$

Q. What happens as the black holes are moved closer together?

A. Transition from two distinct systems to a composite system:
   - For $a > a_2$, saddle points of $h$ are above and below equatorial plane
   - For $a < a_1$, saddle points of $h$ are in the plane
   - For $a_1 < a < a_2$, there is a local maximum of $h \Rightarrow$ Bound orbits!

where $a_1 = 4M/\sqrt{27}$ and $a_2 = \sqrt{2}a_1$. 
Separation of black holes $a$ and critical contours of $h$

- For $a = 1$, three saddle points are connected by a single contour at $p_{\phi} = \frac{1}{2} 5^{5/4} \varphi^{3/2}$, where $\varphi$ is the **Golden Ratio**.

- The saddle points are:
  - in the equatorial plane at $\rho = \frac{1}{2} 5^{1/4} \varphi^{3/2}$
  - above/below the plane at $\rho = \frac{1}{2} 5^{1/4} \varphi^{-1/2}$, $z = \pm 1/(2 \varphi)$.

- There is a maximum in the eq. plane at $\rho = \sqrt{3}/2$ with $p_{\phi} = 9 \sqrt{3}/2$. 
5. Stable bounded null orbits
Stable bounded null orbits (I)

Sam Dolan (Sheffield)
Stable bounded null orbits (II)

Null rays in a ‘pocket’ with three throats for \( a = 1, \ p_\phi < p_\phi^{\text{crit}} \).
2D shadow: $\theta = 90^\circ$, $a = 1$

- The black and orange slices are Cantor-like.
- The pink slice is **highly chaotic** (pocket with throats).
2D shadow: $\theta = 90^\circ$, $a = 1$

Zooming in on the upper fronds.
6. The Lakes of Wada
Lakes of Wada

Brouwer (1910)

- Imagine three or more non-overlapping regions filling a plane, such that any point of the boundary of one region is on the boundary of all regions.

Koneyama (1917)

- Every open neighbourhood of a point on a boundary has a non-empty intersection with at least 3 different basins. (He attributed to T Wada, his supervisor).

Image from CuriosaMathematica: http://curiosamathematica.tumblr.com/
Wada property in Fractal Geometry

- The iteration of a complex function \( f(z) \) generates a dynamical system.
- The **Julia set** \( J(f) \) is the closure of the set of repelling periodic points of \( f \).

**Shared boundaries of basins of attraction**

- **Lemma**: Let \( w \) be an attractive fixed point of \( f \). Then the boundary of the basin of attraction of \( w \) is equal to the Julia set \( J(f) \). The same is true if \( w = \infty \). [Lemma 14.11, K Faulkner “Fractal Geometry” (1990)].

- **Corollary**: All basins of attraction share a common boundary \( J(f) \). Thus, three or more attractive fixed points implies the Wada property.

- **Example**: Newton-Raphson method for \( n \)th roots of unity \( (n \geq 3) \).
Newton-Raphson method: cubic roots of unity

Three basins of attraction

The shared boundary (the Julia set)
The Hénon–Heiles Hamiltonian

- A well-studied 2D Hamiltonian system with **three escapes**:
  \[
  H = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + V(x, y)
  \]
  \[
  V(x, y) = \frac{1}{2} \left( x^2 + y^2 \right) + x^2 y - \frac{1}{3} y^3.
  \]
- The exit basins have been shown to possess the Wada property [Aguirre, Vallejo, Sanjuán 2001].
Basins of attraction

Hénon–Heiles  

Majumdar–Papapetrou
Invariant sets: the strange saddle

Hénon–Heiles

Majumdar–Papapetrou
7. On the (non-)existence of stable bounded null orbits
We found that the Maj.–Pap. spacetime admits stable bounded null orbits for $a_1 < a < \sqrt{2}a_1$. 

This led to qualitatively-different shadow properties.

Q. Are such orbits expected in ‘realistic’ binary BHs? 
A. Probably not.
On the (non-)existence of bounded stable null orbits

‘Theorem’

Stable bounded null orbits are not admitted by static axisymmetric vacuum ($A_\mu = 0$) spacetimes.
On the (non-)existence of bounded stable null orbits

‘Proof’:

1. Weyl ansatz for static axisymmetric electrovacuum spacetime:

\[ ds^2 = -U^{-2}dt^2 + U^2 \left( e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2 \right), \quad A = [A_0, 0]. \]

2. Einstein-Maxwell equations:

\[ U \nabla^2 U = (\nabla U)^2 - (U^2 \nabla A_0)^2, \quad \nabla (U^2 \nabla A_0) = 0. \]

3. Null geodesic equations in Hamiltonian form \((H = 0)\):

\[ H = \frac{1}{2} \left( p^2_\rho + p^2_z \right) + \frac{1}{2\rho^2} e^{2\gamma} (h + p_\phi) (h - p_\phi), \quad h \equiv \rho U^2. \]

4. Classify the stationary points of \(h\) to understand null geodesic structure.
On the (non-)existence of bounded stable null orbits

‘Proof’:

4. Classify the stationary points of $h$.
   Use field equations to show that, at a SP,
   \[
   h_{,\rho\rho} + h_{,zz} = -2\rho \left( U^2 \nabla A_0 \right)^2.
   \]

5. **In vacuum** ($\nabla A_0 = 0$)
   - $h_{,\rho\rho} = -h_{,zz} \Rightarrow$ SP discriminant $\Delta \leq 0$:
     \[
     \Delta \equiv h_{,\rho\rho} h_{,zz} - h^2_{,z\rho} = -(h^2_{,zz} + h^2_{,z\rho}) \leq 0
     \]
   - $\Delta < 0 \Rightarrow$ stationary point of $h$ is a saddle point $\Rightarrow$ unstable null orbit.
   - The marginal case $\Delta = 0$ needs further attention to complete the proof.

6. **In electrovacuum**, the discriminant is negative if $h_{,\rho\rho}$ is positive.
   Thus local minima are forbidden. However, local maxima are not forbidden by this argument.
On the (non-)existence of bounded stable null orbits

Extremally-charged (Maj–Pap)  Uncharged (Weyl-Bach)
On the (non-)existence of bounded stable null orbits

**Further remarks:**

- In the Maj-Pap spacetime, in a certain regime, $h$ admits a maximum – and thus stable null orbits are possible. By contrast, stable null orbits are forbidden in the *vacuum* axisymmetric static case.

- I anticipate that stable null orbits are also permitted in a wide subclass of charged Bretón–Manko–Aguilar (1998) spacetimes.

- The BMA family includes the Weyl–Bach and Maj.–Pap. diholes as special cases.

- The BMA family includes diholes of non-extremally-charged Reissner-Nordström BHs, held in position with Weyl struts.

- **However** a real binary BH is neither static nor axisymmetric ($p_t \neq \text{const.}$ and $p_\phi \neq \text{const.}$), and is unlikely to be significantly far from vacuum.
Conclusions

- Binary BHs exhibit **chaotic scattering**: a fractal set of scattering singularities at which the deflection angle diverges.

- Black hole shadows naturally acquire a **Cantor-like structure**, due to transitions between distinct null orbits.

- Conversely, where transitions are forbidden, the shadow is **regular**.

- The Majumdar–Papapetrou di-hole is an interesting-but-flawed binary surrogate:
  - It admits **stable null orbits** for $\sqrt{16/27} < a/M < \sqrt{32/27} \ldots$
  - ...creating qualitatively-different shadows (Wada?) but ...
  - stable null orbits are **unlikely** in real binary BHs.

- **To Do**: Let’s investigate ...
  - toy models for binaries (e.g. BMA, Kastor-Traschen solutions)
  - stationary axisymmetric models with rotation.
  - models of tidally-distorted BHs; hairy BHs and bumpy BHs.
  - numerical spacetimes via ray-tracing (cf. Bohn et al. 2015).
Selected references

  “Binary black hole shadows, chaotic scattering and the Cantor set”

- A. Bohn, W. Throwe, F. Hébert et al.,
  “What does a binary black hole merger look like?”,


- C. Dettmann, N. Frankel and N. Cornish, Fractals 3, 161 (1995) [arXiv:gr-qc/9502014]
