

# MAS212 Scientific Computing and Simulation

#5: Numerical methods for ODEs: (a) explicit methods

<http://sam-dolan.staff.shef.ac.uk/mas212>

## Key resources:

- Lec 5: <http://sam-dolan.staff.shef.ac.uk/mas212/docs/15.pdf>
- [http://en.wikipedia.org/wiki/Numerical\\_methods\\_for\\_ordinary\\_differential\\_equations](http://en.wikipedia.org/wiki/Numerical_methods_for_ordinary_differential_equations)

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import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

## 1. Difference equations. Consider the difference equation

$$x_{k+1} = x_k + 2h\sqrt{x_k} \quad (\dagger)$$

and  $t_{k+1} = t_k + h$ .

(a) Let  $h = 0.1$ . Use  $(\dagger)$  to find  $x_k$  and  $t_k$  for  $k = 0 \dots 20$ , starting with  $t_0 = 0$  and  $x_0 = 0.2$ . Plot  $x_k$  against  $t_k$ . Now add lines on the plot for initial conditions  $x_0 = 0.4$ ,  $x_0 = 0.6$ ,  $x_0 = 0.1$  and  $x_0 = 0$ . (At this point you may wish to write a function which takes  $h$  and  $x_0$  as parameters, and returns arrays of  $t$  and  $x$  values).

(b) Eq.  $(\dagger)$  is obtained by applying **Euler's method** to a first-order ODE. Deduce the corresponding ODE and solve it analytically by separation of variables. Compare the exact solutions with your numerical solutions (and comment on the  $x_0 = 0$  numerical solution.)

(c) **The ratio test.** For  $x_0 = 0.2$ , obtain three data sets: (i)  $h = 0.1$  and  $k = 0 \dots 20$ ; (ii)  $h = 0.05$  and  $k = 0 \dots 40$ ; (iii)  $h = 0.025$  and  $k = 0 \dots 80$ . Plot the three solutions. Which is most accurate, and why? Now compute the ratio:

$$r_k = \frac{x_k^{(i)} - x_{2k}^{(ii)}}{x_{2k}^{(ii)} - x_{4k}^{(iii)}}, \quad k > 0.$$

Plot  $r_k$  against  $t_k^{(i)}$ . What do you observe? (Now try halving all values of  $h$ , and plot again).

(d) Try changing the  $\sqrt{x_k}$  term in  $(\dagger)$  to something else and repeat step (c). Why is the ratio approximately constant, and why does it take this particular value?

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## 2. The midpoint method.

Write a function to implement the midpoint method

$$\begin{aligned} x_{k+1/2} &= x_k + \frac{1}{2}h f(x_k, t_k), \\ x_{k+1} &= x_k + h f(x_{k+1/2}, t_{k+1/2}), \end{aligned}$$

and use it to obtain a numerical solution of the ODE  $\frac{dx}{dt} = \sin^3(t)$  with  $x(0) = -1$ .

(a) Plot the solution with  $h = 0.1$  across the domain  $t \in [0, 20]$ , i.e. for the sequence of values  $t_k = kh$ .

(b) Apply the ratio test, as in 1(c). What do you find? What happens when  $x$  passes through zero?

(c) Now try solving  $\frac{dx}{dt} = \sin^n(t)$  where  $n$  is large. What do you observe? How small does  $h$  have to be to resolve the key features?

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**3. A second-order equation.** The van der Pol equation may be written in first-order form as

$$\dot{x} = y, \quad \dot{y} = \mu(1 - x^2)y - x.$$

Show that application of Euler's method to these equations leads to the difference equations

$$\begin{aligned}x_{k+1} &= x_k + hy_k \\y_{k+1} &= y_k + h(\mu(1 - (x_k)^2)y_k - x_k).\end{aligned}$$

where  $t_{k+1} = t_k + h$ .

(a) Write a function to solve the van der Pol equation with Euler's method. For  $\mu = 1$  and a variety of initial conditions, plot of  $x$  against  $y$  (i.e. make a phase plot). Can you resolve the limit cycles using Euler's method, or is a more sophisticated method required?

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**4. The third-order Runge-Kutta method.**

Suppose we have an ODE of the form  $\frac{dx}{dt} = f(x, t)$ . Kutta's third-order method is:

$$x_{n+1} = x_n + \frac{1}{3}h(k_1 + k_2 + k_3)$$

where

$$\begin{aligned}k_1 &= f(x_n, t_n) \\k_2 &= f\left(x_n + \frac{1}{4}hk_1, t_n + \frac{1}{4}h\right) \\k_3 &= f\left(x_n - \frac{2}{3}hk_1 + hk_2, t_n + \frac{1}{3}h\right)\end{aligned}$$

Implement Kutta's method, and use it to solve the equation

$$\frac{dx}{dt} = 2\sqrt{x} + \sin(t).$$

Apply the ratio test to show that the method is third-order-accurate.

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**Answers:** 1(b)  $\frac{dx}{dt} = 2\sqrt{x}$ , with solution  $x(t) = (t + \sqrt{x_0})^2$ , where  $x_0$  is a constant. NB The numerical solution for  $x_0 = 0$  is not what we might expect – why? 1(c) The ratio is  $\sim 2$ . As  $h \rightarrow 0$ , the ratio should approach 2. 1(d) The value of the ratio is a property of the method, rather than the particular ODE chosen. The ratio is  $r \sim 2^b$  where  $b$  is the order of accuracy of the method (meaning that the global truncation error scales with  $h^b$ ). Hence  $r \sim 2$  implies that Euler's method is only first-order accurate.