MAS212 Scientific Computing and Simulation

#4: Solving ODEs with `scipy.integrate.odeint`

`http://sam-dolan.staff.shef.ac.uk/mas212`

Key resources:

- Lec 4: `http://sam-dolan.staff.shef.ac.uk/mas212/docs/l4.pdf`

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
%matplotlib inline   # include plots in Jupyter notebook
```

1. First-order Ordinary Differential Equations (ODEs).

(a) Let

$$\frac{dx}{dt} = -x, \qquad x(0) = 1.$$

(i) Write a function `dxdt(x, t)` that returns the derivative (`-x`). Solve the ODE using `odeint`, and plot over the domain $0 \le t \le 5$.

(ii) By hand, use the method of separation of variables to show that the general solution is $x(t) = x_0 e^{-t}$ (with $x_0 = 1$ in this case).

(iii) Compare the exact and numerical solutions on a plot.
Calculate the error – the difference between the numerical solution from `odeint` and the exact solution – on the same domain. How could the error be reduced further? (*Hint:* Set tolerances with the optional arguments `atol` and `rtol` of `odeint`).

(b) Now consider the **logistic equation**,

$$\frac{dx}{dt} = x(1-x), \qquad x(0) = x_0.$$

Plot the solutions for various initial conditions x_0 in the range 0 to 2. Identify the stable and unstable equilibria.

(c) Let's go one step further,

$$\frac{dx}{dt} = -\frac{1}{2}(x-1)x(x+1), \qquad x(0) = x_0.$$

Plot solutions for initial conditions $x_0 = [-3, -2.75, -2.5, \ldots, 2.75, 3]$. Identify the stable and unstable equilibria.

2. The Predator-Prey Equations.

The predator-prey (Lotka-Volterra) equations represent a simplified model of the change in populations of two species which interact via predation. For example, foxes (predators) and rabbits (prey). Let x and y represent rabbit and fox populations, respectively. Then

$$\frac{dx}{dt} = ax - bxy, \qquad \frac{dy}{dt} = -cy + dxy, \qquad (a, b, c, d > 0).$$

(a) Solve the equations with $a = b = c = d = 1$ using `odeint()` for a variety of initial conditions, and make a labelled plot of the populations of rabbits and foxes as a function of time.

(b) Make a phase plot of x vs y (rabbits vs foxes) as described in the lecture notes.

(c) Investigate how the curves on the phase plot are modified as b (the appetite of the foxes!) is varied.

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3. Second-order equations.

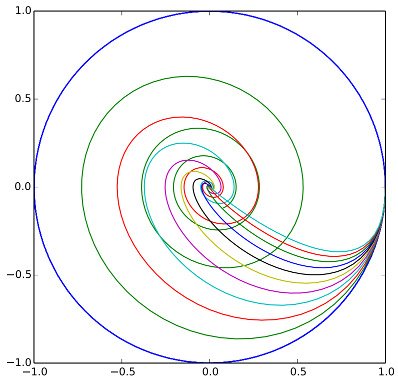
Consider the ODE describing a damped oscillator,

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + x = 0, \qquad x(0) = 1, \ \frac{dx}{dt}(0) = 0,$$

where $\gamma > 0$.

(a) By defining $y = \frac{dx}{dt}$, write down a pair of coupled first-order equations and their initial conditions.

(b) Solve these ODEs for a selection of parameters γ in the range 0 to 2. Make phase plots of x vs y. Describe a key difference between the case $\gamma = 0$ and $\gamma > 0$. In the latter case, $x = 0 = y$ is called a **limit point** – why? Is it attractive or repulsive?



4. van der Pol equation.

$$\ddot{x} - a(1 - x^2)\dot{x} + x = 0, \qquad a > 0.$$

(a) Convert to two first-order ODEs, as described in Question 3.

(b) Plot the **limit cycles** in the phase space for various $0 < a \leq 3$, as shown in the lectures.

(c) Investigate the limit cycles (if they exist) for the *forced* van der Pol equation,

$$\ddot{x} - a(1 - x^2)\dot{x} + x = b \sin \omega t, \qquad a > 0$$

for various values of b and ω.