MAS212 Assignment #2: The damped driven pendulum

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1 Introduction

In this assignment we study the motion of a rigid pendulum of length ℓ and mass m, shown in Fig. 1, using both analytical and numerical methods. The state of the system is described by the angle θ of the rod to the vertical, with $\theta = 0$ corresponding to the equilibrium position where the bob is directly below the pivot. The bob can 'swing over the top'.

Early time-keeping devices, such as grandfather clocks, used pendulums to mark time, relying on their regular oscillations. At small amplitudes ($\theta \ll 1$), the motion is regular with a fixed period. However, the *driven* damped oscillator, studied in Sec. 3, is one of the simplest systems to show chaotic dynamics [1], that is, irregular and unpredictable motion with an exponential sensitivity to initial conditions [2].

(a) The damped harmonic oscillator.

The bob of the pendulum is subject to three forces: a normal force along the rod F_N, the force of gravity acting downwards, F_g and a frictional force F_f, acting opposite to the direction of motion and proportional to $\dot{\theta}$. These forces produce a torque of $G = -\ell(mg\sin\theta + \mu\dot{\theta})$ [3], which leads to an acceleration of $\ddot{\theta} = G/I$, where $I = m\ell^2$ is the moment of inertia. Rearranging,

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2 \sin\theta = 0, \tag{1}$$

where $2\gamma = \mu/\ell m$ and $\omega^2 = g/\ell$.

The parameter ω is the natural angular frequency of the system. In the absence of friction ($\gamma = 0$), small-amplitude displacements ($\theta \ll 1$) lead to simple harmonic motion with a period $T = 2\pi/\omega$. The parameter γ is associated with damping. For $0 < \gamma < \omega$, the oscillations decrease in amplitude over time, with a half-life of $\tau = \ln 2/\gamma$.

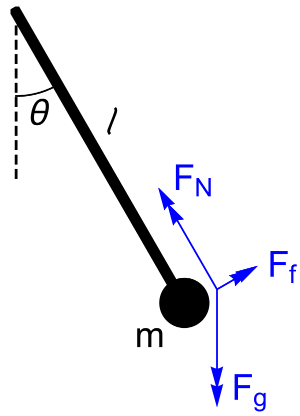

Figure 1: Pendlum rod.

(b) Linearization: If the pendulum is near the vertical then $\theta \ll 1$ and we may use a Maclaurin series expansion, $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \ldots$. The terms in this series get successively smaller, since $\theta \gg \theta^3 \gg \theta^5$, etc. Using the approximation $\sin\theta \approx \theta$ leads to a linear equation,

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2\theta = 0. \tag{2}$$

Superposition: As Eq. (2) is linear and homogeneous, any superposition of solutions yields another solution. For example, let $\theta_3(t) = A\theta_1(t) + B\theta_2(t)$, where A and B are constants, and $\theta_1(t)$ and $\theta_2(t)$ satisfy Eq. (2). It follows that

$$\begin{aligned}\ddot{\theta}_3 + 2\gamma\dot{\theta}_3 + \omega^2\theta_3 &= \left(A\ddot{\theta}_1 + B\ddot{\theta}_2\right) + 2\gamma\left(A\dot{\theta}_1 + B\dot{\theta}_2\right) + \omega^2\left(A\theta_1 + B\theta_2\right), \\ &= A\left(\ddot{\theta}_1 + 2\gamma\dot{\theta}_1 + \omega^2\theta_1\right) + B\left(\ddot{\theta}_2 + 2\gamma\dot{\theta}_2 + \omega^2\theta_2\right) \quad = 0. \end{aligned} \tag{3}$$

Thus, $\theta_3(t)$ is also a solution of Eq. (2).

Equation (2), which produces damped harmonic motion, appears in several other contexts. For example, the charge Q flowing in a closed electrical circuit with a resistor (of resistance R), capacitor (of capacitance C) and an inductor (of inductance I) is governed by the equation

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0. \tag{4}$$

This is mathematically equivalent to Eq. (2) under the mapping $\theta \leftrightarrow Q$, $\gamma \leftrightarrow R/2L$ and $\omega^2 \leftrightarrow 1/LC$.

(c) Exact solution. Equation (2) has an exact solution given by [4]

$$\theta(t) = Ae^{-\gamma t}\cos\left(\sqrt{\omega^2 - \gamma^2}\, t + \delta\right). \tag{5}$$

For brevity, let $c \equiv \cos(\alpha t + \delta)$ and $s \equiv \sin(\alpha t + \delta)$, with $\alpha \equiv \sqrt{\omega^2 - \gamma^2}$. One may verify that (5) is a solution of Eq. (2) using back-substitution. Differentiating, $\dot{\theta} = Ae^{-\gamma t}(-\gamma c - \alpha s)$ and $\ddot{\theta} = Ae^{-\gamma t}(\gamma^2 c + 2\alpha\gamma s - \alpha^2 c)$. Inserting into Eq. (2) yields

$$\begin{aligned} &Ae^{-\gamma t}\left[\left(\gamma^2 c + 2\alpha\gamma s - \alpha^2 c\right) + 2\gamma\left(-\gamma c - \alpha s\right) + \omega^2 c\right] \\ &= Ae^{-\gamma t}\left[\left(2\alpha\gamma - 2\alpha\gamma\right)s + \left(\omega^2 - \gamma^2 - \alpha^2\right)c\right] = 0. \end{aligned} \tag{6}$$

Note here that $\alpha^2 = \omega^2 - \gamma^2$, from the definition of α above.

The exact solution for $\gamma > \omega$ is $\theta(t) = Ae^{-(\gamma+\beta)t} + Be^{-(\gamma-\beta)t}$, where $\beta = \sqrt{\gamma^2 - \omega^2}$ and A, B are real constants. The exact solution for $\gamma = \omega$ is $\theta(t) = (A + Bt)e^{-\gamma t}$.

An *underdamped* system ($\omega > \gamma$) will undergo damped oscillations about its equilibrium point ($\theta = 0$). An *overdamped* system ($\gamma > \omega$) will approach the equilibrium without oscillating. A *critically-damped* system will approach the equilbrium as quickly as possible without over-shooting [4]

2 The free pendulum

(a) First-order reduction. By introducing a new variable $\eta \equiv \dot{\theta}$, and noting that $\ddot{\theta} = \dot{\eta} = -2\gamma\dot{\theta} - \omega^2\sin\theta$, by Eq. (1), one can write Eq. (1) as a pair of first-order ordinary differential equations,

$$\dot{\theta} = \eta, \qquad \dot{\eta} = -2\gamma\eta - \omega^2\sin\theta. \tag{7}$$

(b) Example solution. I used the `odeint()` function in the `scipy.integrate` module of Python, to compute an example solution of Eq. (7), with parameters $\gamma = 0.1$, $\omega = 1$, and initial conditions $\theta(0) = \pi/2$ and $\eta(0) = 0$. The example solution is shown in Fig. 2 in red.

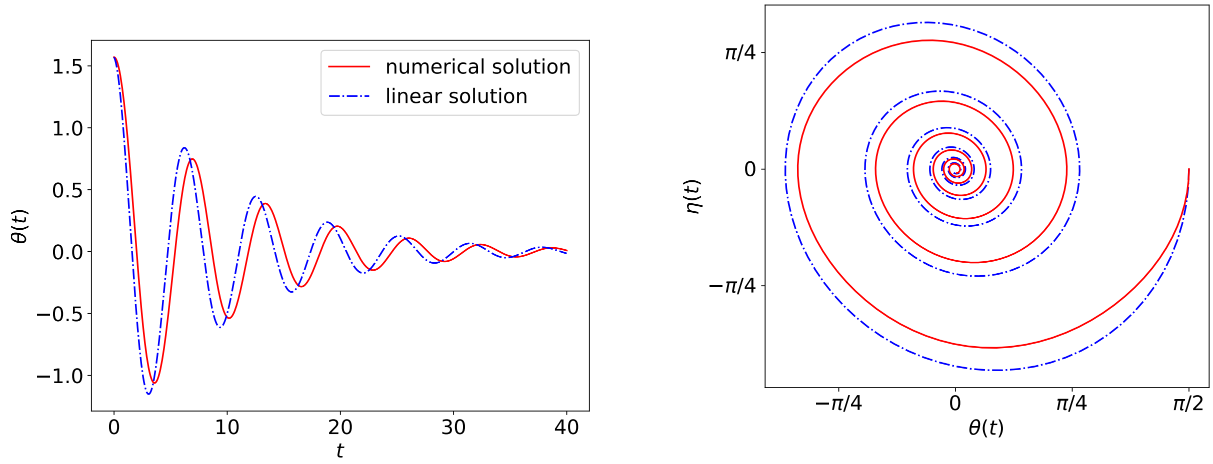

Figure 2: *Left:* a time-domain plot, showing the deflection angle of the pendulum $\theta(t)$ released from rest at $\theta(0) = \pi/2$, with parameters $\gamma = 0.1$ and $\omega = 1$. The numerical solution (red, solid) of Eq. (7) is compared with the exact solution to the linearized equation (blue, dashed). *Right:* a phase portrait of θ versus $\eta \equiv \dot{\theta}$. The trajectories spiral towards the fixed point at $(0, 0)$.

The left plot shows how $\theta(t)$ and $\eta(t)$ undergo a damped oscillation over time. The right plot shows a phase portrait in the (θ, η) plane, showing that the trajectory spirals towards a fixed point at $(0, 0)$.

The plot also shows the exact solution of the linearized equation with the same initial conditions, that is, Eq. (5) with $\tan\delta = -\gamma/\alpha$ where $\gamma = 0.1$, $\alpha = \sqrt{\omega^2 - \gamma^2} \approx 0.9950$, $\delta \approx -0.1002$, $A \approx 1.5787$. Initially, the solutions (red and blue) move apart, as the large initial amplitude means that Eq. (2) is a poor approximation of Eq. (1). Once the amplitude has decreased somewhat, both solutions oscillate with approximately the same frequency.

(c) Fixed point. The ODE system (7) has a fixed point at $(\theta = 0, \eta = 0)$, where $\dot{\theta} = 0$ and $\dot{\eta} = 0$. The Jacobian matrix J at $(0, 0)$ is

$$J(\theta, \eta) = \begin{pmatrix} \frac{\partial \dot{\theta}}{\partial \theta} & \frac{\partial \dot{\theta}}{\partial \eta} \\ \frac{\partial \dot{\eta}}{\partial \theta} & \frac{\partial \dot{\eta}}{\partial \eta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos\theta & -2\gamma \end{pmatrix} \quad \Rightarrow \quad J(0, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\gamma \end{pmatrix}. \tag{8}$$

The trace and determinant of J are $\tau = -2\gamma$ and $\delta = \omega^2$, respectively. The trace is negative and $\tau^2 - 4\delta = 4(\gamma^2 - \omega^2)$ is negative in the under-damped case; this implies that the eigenvalues of J are complex with negative real parts, and thus the fixed point is a *spiral sink*. In the over-damped and critically-damped cases, the eigenvalues of J are real and negative, and thus the fixed point is a *sink node*.

3 The driven non-linear pendulum

In this section we consider the effect of a periodic driving force on the motion of the pendulum, by solving the ODE

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2 \sin\theta = \alpha\cos(\Omega t), \tag{9}$$

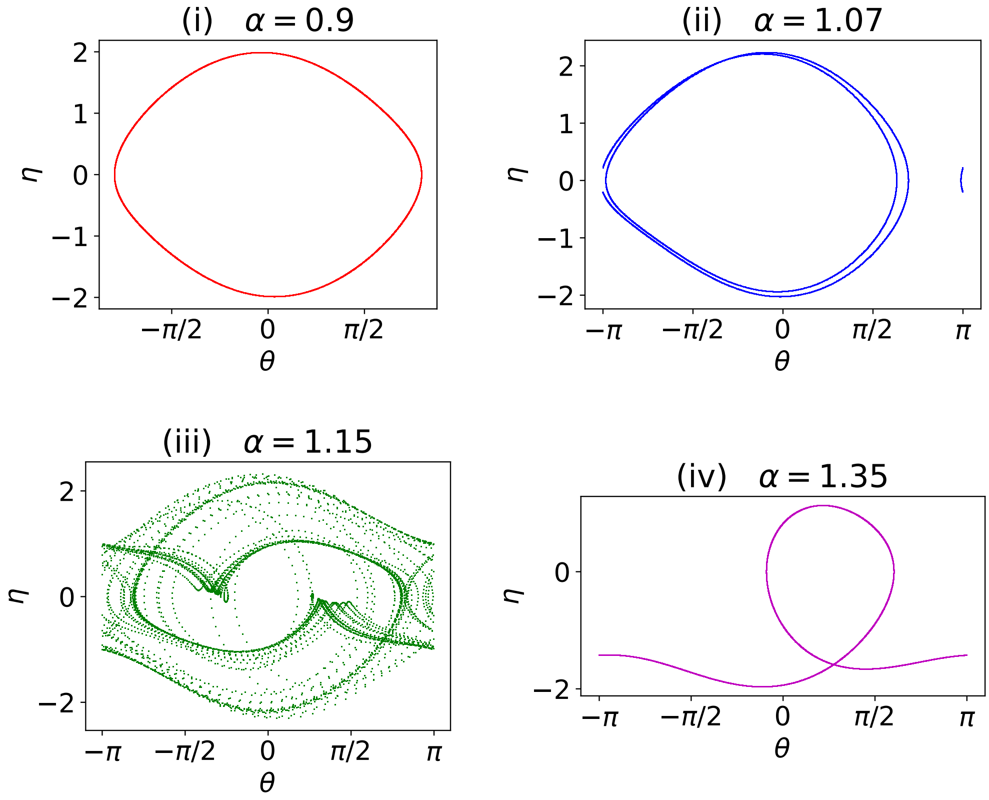

Figure 3: Phase portraits in the(θ, η) plane for cases (i)–(iv) described in the text.

or, in first-order autonomous form,

$$\dot{\theta} = \eta, \quad \dot{\eta} = -2\gamma\eta - \omega^2 \sin\theta + \alpha\cos\varphi, \quad \dot{\varphi} = \Omega. \tag{10}$$

We seek to address a key question: after initial transient behaviour, does the pendulum swing in a periodic fashion, with a period which is a multiple of the period of the driving term, $T = 2\pi/\Omega$? Or is the response aperiodic?

(a) Phase plots. Figure 3 shows examples of the response of the system to the driving force in the (θ, η) plane, with parameters $\gamma = 1/4$, $\omega = 1$, $\Omega = 0.6667$ and (i) $\alpha = 0.9$, (ii) $\alpha = 1.07$, (iii) $\alpha = 1.15$ and (iv) $\alpha = 1.35$. In all cases, I started the pendulum from rest at the equilibrium position (i.e. $\theta = 0, \eta = 0, \varphi = 0$ at $t = 0$). I have plotted only the late-time (non-transient) behaviour, for $500 \le t \le 1000$. The angle θ is shown in the range $-\pi \le \theta \le \pi$.

For $\alpha = 0.9$, the pendulum follows a closed loop in phase space, implying periodic motion. For $\alpha = 1.07$, there is a "double-loop", and on every other swing the pendulum moves just over the top (past the vertical), before swinging down again. For $\alpha = 1.35$, the pendulum swings over the top every time, moving around in a counter-clockwise direction, in a periodic fashion. For $\alpha = 1.15$, the response is qualitatively different. It appears that the pendulum motion does not repeat itself, although the motion is not completely random. The phase portrait shows some structure; for example, the trajectory does not pass close to $\theta = \eta = 0$,

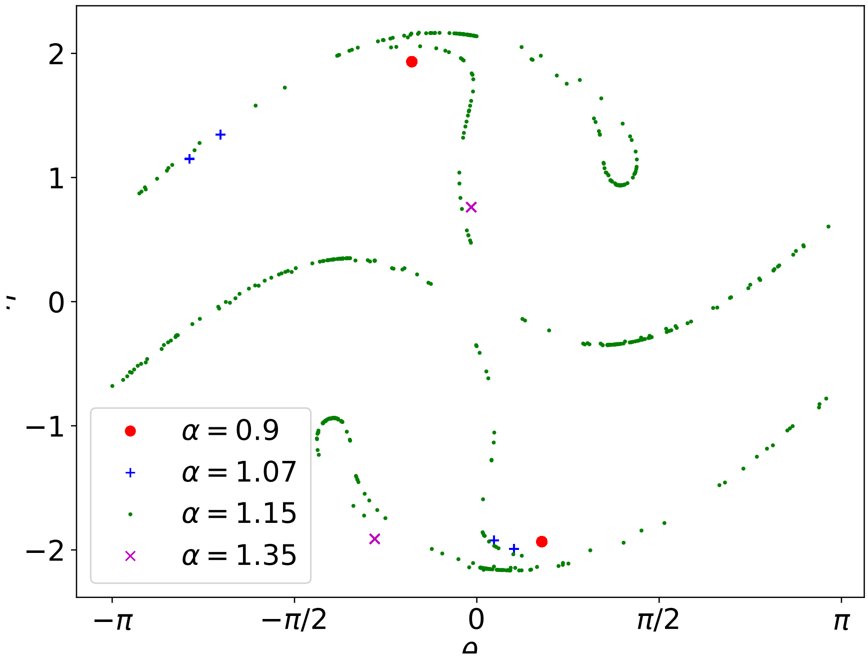

Figure 4: A stroboscopic plot for cases (i)–(iv) described in the text, showing how the trajectories in the 3D phase space intersect the submanifold on which $\sin\phi = 0$.

whereas it 'fills up' other parts of the phase space. In cases (i), (ii) and (iv) the response is periodic; in case (iii) it is aperiodic.

(b) Stroboscopic plots. A complementary way to visualize these motions is through a stroboscopic plot (also known as a Poincaré section). Figure 4 shows the values of (θ, η) at times $n_1 T, (n+1)T, \ldots, n_2 T$, where $T = \pi/\Omega$ is half the period of the driving force. These are the times at which $\sin\varphi = 0$. In cases (i), (ii) and (iv) a repeating pattern appears, with just 2, 4 and 2 points visible, respectively. This implies that in cases (i) and (iv) the pendulum has the same period as the driving force, and in case (ii) it has twice that period. Case (iii) is different; the points do not repeat, but instead trace out a structure in phase space. Close inspection shows that the structure has detail on fine scales. In this case, the motion is not periodic.

(c) Extensions. Figure 5 shows phase portraits in the 3D phase space (θ, η, φ). It shows that the curves do not cross in the 3D phase space (as anticipated in an autonomous system). In case (iii), the plot shows interesting 3D structure. Close inspection suggests that the system is often 'close to' a repeating cycle, but it also 'wanders off' into other parts of the phase space.

When is the response aperiodic? To investigate this I tried varying the parameter α. Figure 6 shows the values of θ in the stroboscopic plot for each value of α, keeping all other parameters fixed. It shows that regular and chaotic behaviours are present, depending on α. It also shows 'period-doubling' behaviour, as the system moves from order into chaos near $\alpha = 1$.

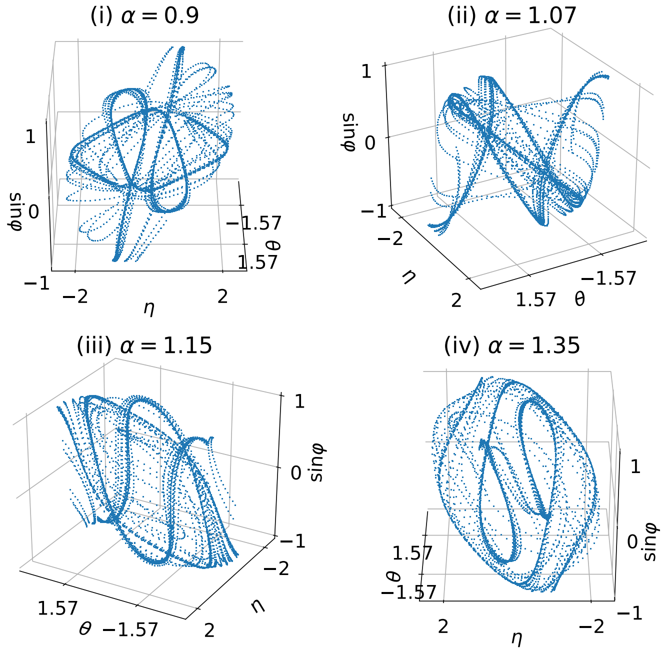

Figure 5: Showing the same trajectories as Fig. 3 in the 3D phase space.

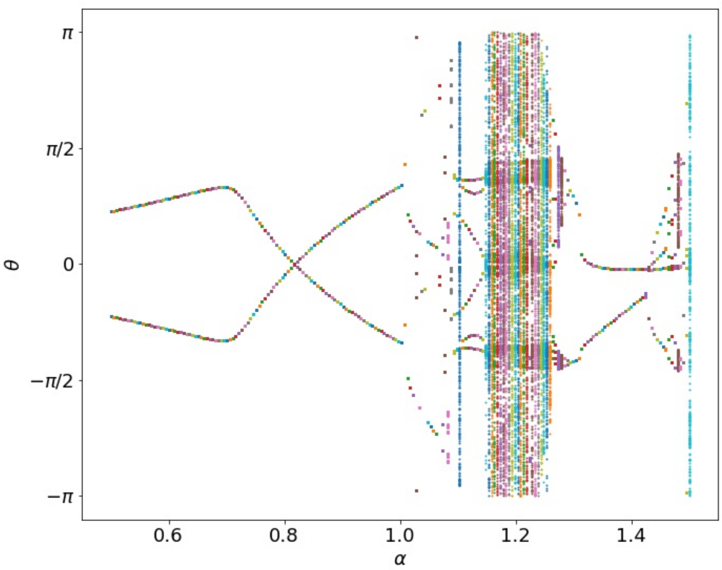

Figure 6: Bifurcation diagram. Showing the values of θ in the Poincaré section with $\sin\varphi = 0$, as a function of driving amplitude α.

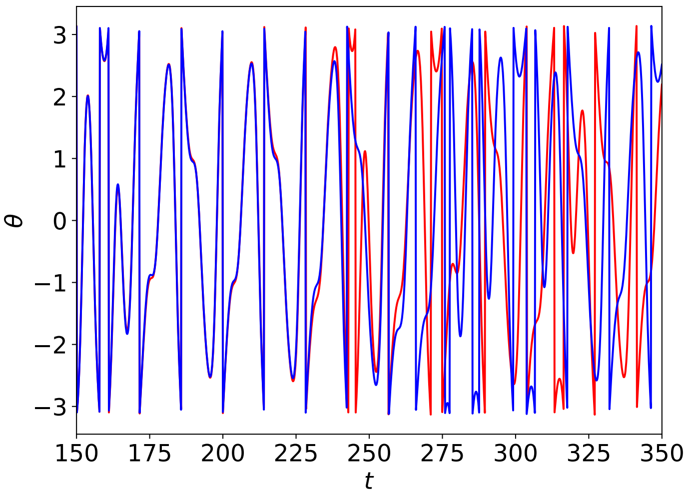

Figure 7: Sensitivity to initial conditions. The two lines show the motion of two identical pendulums with $\alpha = 1.15$, starting with initial conditions $\eta = \varphi = 0$ and (i) $\theta = 0$ and (ii) $\theta = 1 \times 10^{-8}$. Up to $t \sim 250$ the two systems are close together, by eye, but at later times the two systems behave differently, due to the extreme sensitivity of a chaotic system to initial conditions.

Imagine starting the pendulum twice, from slightly different positions. In cases (i), (ii) and (iv), an initially-small difference with grow with time, but in a linear fashion. By contrast, for case (iii) a small difference can grow exponentially. Figure 7 shows that a difference in initial conditions of just 10^{-8} eventually leads to the two pendulums behaving very differently. As small errors are inevitable when measuring the initial state of a system, this behaviour implies the break down of predictability for chaotic systems, at least beyond a certain time (related to the Lyapunov exponent). This is known as *deterministic chaos*.

Summary: We have investigated some of the rich phenomenology of the driven damped pendulum, showing that it can oscillate in a regular (periodic) or irregular (chaotic) manner, depending on the system parameters. In the latter case, the system exhibits an extreme sensitivity to initial conditions. In 1972, Edward Lorenz noted this sensitivity in another non-linear system when modelling the weather, leading him to pose a famous question: "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

References

[1] Baker, G. L., and Gollub, J. P., *Chaotic Dynamics* (Cambridge University Press, 1996, 2nd ed.)

[2] Goldstein, H., Poole, C. P., and Safko, J. L., *Classical Mechanics* (Pearson, 2001, 3rd ed.)

[3] Dolan, S. R., *Assignment #2 Background*, MAS212 course website (2017) `http://sam-dolan.staff.shef.ac.uk/mas212/docs/assign2_background.pdf`.

[4] Riley, K. F., Hobson, M. P., and Bence, S. J., *Mathematical methods for physics and engineering* (Cambridge University Press, 2006, 3rd ed.)

[5] Example code available here: `http://sam-dolan.staff.shef.ac.uk/mas212/docs/Assign2_Code.ipynb`