MAS212 Assignment #2:
The damped driven pendulum

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Introduction: In this assignment you will use Python to investigate a non-linear differential equation which models the motion of a damped, driven pendulum rod. Part 2 focusses on the *linear* case, describing small-amplitude oscillations. Part 3 focusses on the *non-linear* case, which exhibits some interesting behaviour.

The Submission: A completed assignment will comprise a `.pdf` of your report and a single code file (`.py` or `.ipynb`). Your report should comprise 3 sections corresponding to the 3 parts below. It should also include a conclusion paragraph, and a list of references in a standard academic style. As well as addressing the parts of the brief, your report should also be a coherent document in its own right, which could be read by someone who has not seen this brief. The report should be no more that **eight sides** including figures. If necessary, additional material can be included in an appendix. The report should be accompanied by *one* Python script or Jupyter notebook. Note that the report will carry *much* more credit ($\sim 80\%$) than the code. Please use LaTeX to write your report (or discuss with the lecturer if not possible).

The Deadline: The deadline for submission is found on the course website. Files should be submitted at `http://somas-uploads.shef.ac.uk/mas212`.

Part 1: Introduction

This section should consist of text and mathematics. No code is required for this section. Please include three or more citations of appropriate literature (e.g. scientific papers, books or appropriate online resources).

(a) Damped harmonic oscillator. Write paragraph(s) to introduce the non-linear damped harmonic oscillator equation

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2 \sin\theta = 0, \qquad \gamma \geq 0, \quad \omega > 0, \tag{1}$$

where $\dot{\theta} = \frac{d\theta}{dt}$, $\theta = \theta(t)$. Describe how this arises as the equation of motion for a pendulum rod (refer to the handwritten notes). Give an interpretation of the two parameters γ and ω.

(b) Linearization and superposition. Eq. (1) is non-linear. Show that, if $|\theta| \ll 1$, we may use the approximation $\sin\theta = \theta + O(\theta^3)$ to obtain a linear equation,

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2\theta = 0, \tag{2}$$

Show that if $\theta_1(t)$ and $\theta_2(t)$ are solutions to Eq. (2), then $\theta_3(t) \equiv A\theta_1(t) + B\theta_2(t)$, where A and B are constants, is also a solution of Eq. (2). Briefly describe one other context which give rises to a second-order equation similar to (2). For example, an LCR circuit or a mass on a spring.

(c) Exact solution. Using substitution, or another method, show that the linearized version of Eq. (1) has an exact solution

$$\theta(t) = Ae^{-\gamma t} \cos\left(\sqrt{\omega^2 - \gamma^2}\, t + \delta\right) \tag{3}$$

for $\omega > \gamma$, where A and δ are constants determined by the initial conditions. Find the exact solution for $\gamma > \omega$. Define and use the terms "under-damped", "over-damped" and "critically-damped".

Part 2: The undriven pendulum

In this section, investigate Eq. (1) by using Python code to compute and plot numerical solutions.

(a) First-order reduction. Show that Eq. (1) can be rewritten as a pair of first-order equations

$$\dot{\theta} = \eta, \qquad \dot{\eta} = -2\gamma\eta - \omega^2 \sin\theta. \tag{4}$$

(b) Example solution. Use `odeint()` to find a numerical solution to (4) for initial conditions $\theta(0) = \pi/2$, $\eta(0) = 0$ and parameters $\gamma = 0.1$, $\omega = 1.0$. Plot the numerical solution [red solid line] and the exact solution Eq. (3) [blue dashed line] in two ways: (i) a time domain plot of $\theta(t)$ vs t; (ii) a phase portrait of $\theta(t)$ on the x-axis versus $\eta(t)$ on the y-axis. Your plots should look somewhat similar to Fig. 1. Include your labelled plots in a figure with a caption. Write a paragraph describing how the pendulum behaves after it is released from rest at $\theta = \pi/2$, referring to your figure for evidence.

(c) Fixed point. In the text, show that Eqs. (4) have a fixed point at $(\theta = 0, \eta = 0)$. Classify the fixed point using the eigenvalues of the Jacobian matrix. You may wish to distinguish between the under-damped and over-damped cases.

Part 3: The driven non-linear oscillator

In this section we consider a *driven* pendulum governed by a modified ODE

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2 \sin\theta = \alpha\cos(\Omega t). \tag{5}$$

Show that this is equivalent to a 3-dimensional first-order autonomous system:

$$\dot{\theta} = \eta, \qquad \dot{\eta} = -2\gamma\eta - \omega^2 \sin\theta + \alpha\cos\varphi, \qquad \dot{\varphi} = \Omega. \tag{6}$$

We will focus on the late-time, long-term behaviour of the system, which is more interesting than the 'transient' response at early times to particular initial conditions. **Key question:** Does the pendulum respond to a periodic driving force by oscillating in a periodic (repeating) way? *(More challenging / open-ended)*

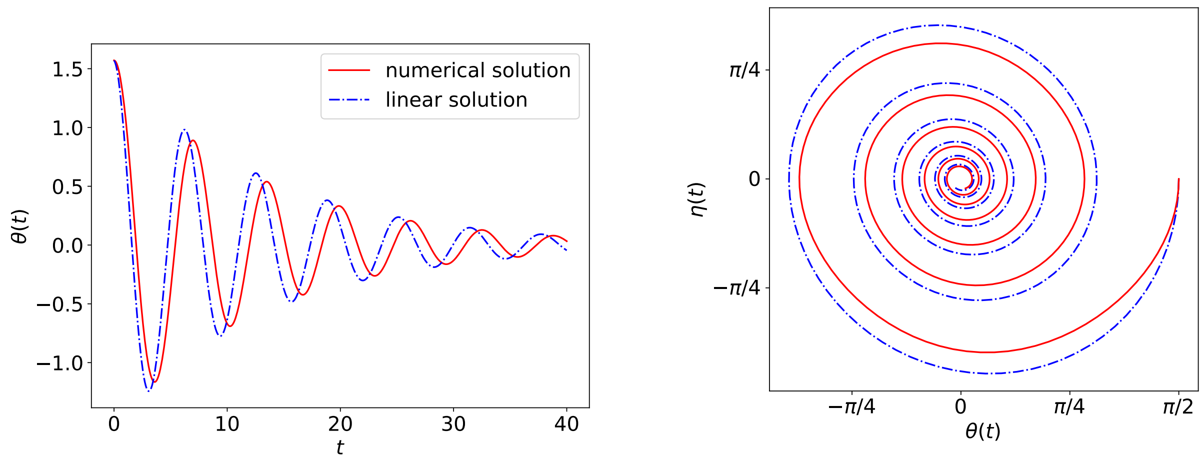

Figure 1: Example plots for part 2(b). *Left:* a time-domain plot, showing the deflection angle of the pendulum $\theta(t)$ as a function of time. *Right:* a phase portrait of θ versus $\eta \equiv \dot{\theta}$. The fixed point at $(0, 0)$. **Warning:** *These plots were generated with different parameters to in part 2(b).*

(a) Phase plots. Make 2D phase plots in the (θ, η) plane for parameter choices $\gamma = 1/4$, $\omega = 1$, $\Omega = 0.6667$ with the following values of α: (i) $\alpha = 0.9$, (ii) $\alpha = 1.07$, (iii) $\alpha = 1.15$, (iv) $\alpha = 1.35$. Use initial conditions $\theta_0 = \eta_0 = \varphi_0 = 0$ at $t = 0$. Focus on the late-time (non-transient behaviour) by plotting for (e.g.) $500 < t < 1000$. Restrict θ to the range $(-\pi, \pi]$. Plot pixels (using `','` option) rather than lines. Your plots should look a bit like those in Fig. 3. Comment on your plots, addressing the key question above.

(b) Stroboscopic plots (Poincaré sections). The phase space is three-dimensional. We may take a 2D cross section of the 3D phase space by plotting a point in the (θ, η) plane every time the integral curve passes through $\sin \varphi = 0$. One may do this by plotting points in the (θ, η) plane at times $n_1 T, (n_1 + 1)T, (n_1 + 2)T, \ldots, n_2 T$ where $T = \pi/\Omega$. (Choose n_1 to be large enough that the initial transient response, relating to the initial conditions, is not visible; for example: $n_1 = 500$, $n_2 = 1000$). Examine the parameter choices of part (a). Is the response periodic? (If there are n distinct points, then the response is periodic with period $\pi n/\Omega$; conversely, if there is not a repeating pattern then the response is not periodic).

(c) Extensions. (Open-ended). Investigate this system thoroughly using Python code. How does its behaviour change as the driving amplitude α is modified? Describe in your own words what happens as α is varied from 0 to 1.5 (for fixed Ω, ω, γ as above). You may illustrate your discussion with additional plots generated by your code. You may wish to consider whether the following concepts are relevant: deterministic chaos; bifurcation; strange attractor.

Conclusion and reference list

Add a short conclusion to your report. This should be four sentences or so, summarising the most important findings. Include a list of references.

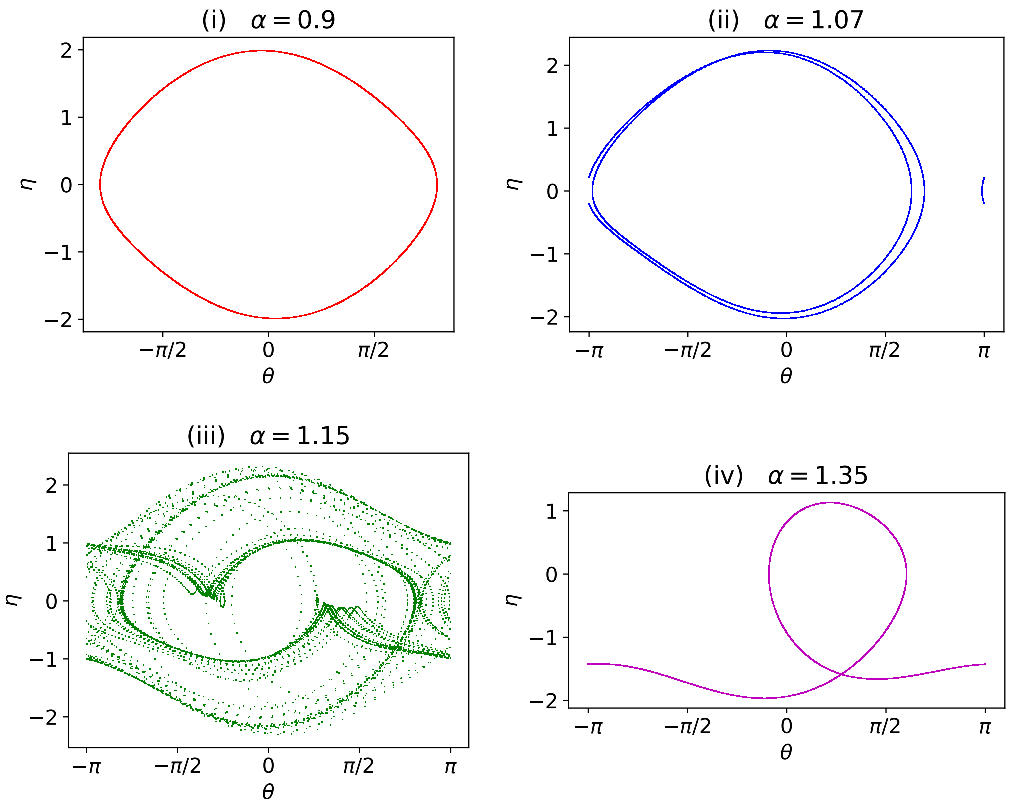

Figure 2: Example plots of (θ, η) for part 3(a).

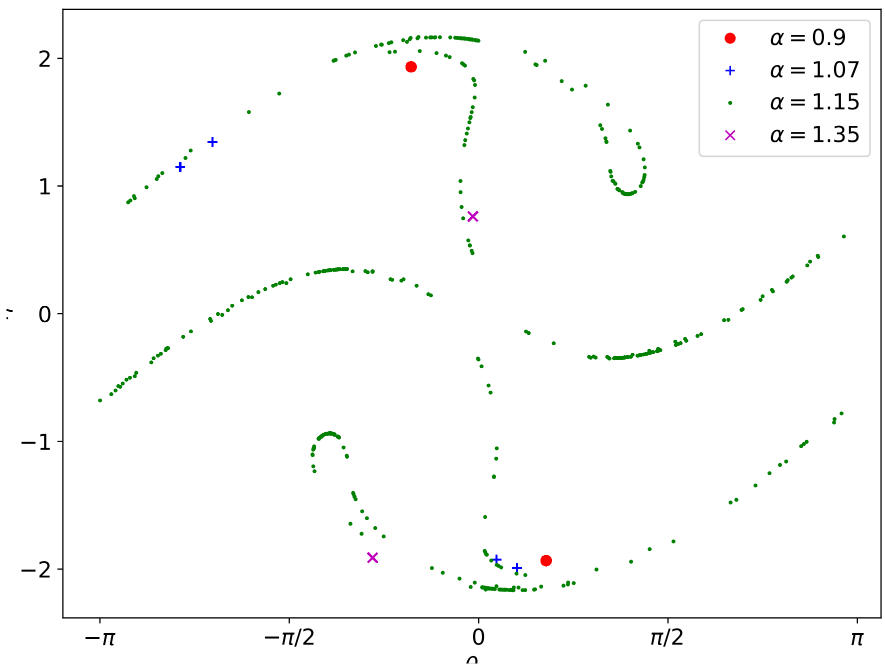

Figure 3: Example of a stroboscopic plot for part 3(b).

Guidance:

This assignment will count for $\sim 30\% - -35\%$ of your module mark, and thus it should be a substantial piece of work. For this assignment, **fair means** include: asking the lecturer for advice; reading online materials; using and adapting short code snippets from lectures; etc. Please avoid verbatim copying, even in the introduction, and please cite all sources used. **Unfair means** include (but are not limited to): sharing or distributing files; copying-and-pasting from work that is not your own; posting your work online; passing off other's work as your own, etc.

Note that all submissions will be checked for excessive similarities. Where there is circumstantial evidence of unfair means, I reserve the right to award zero marks for this assignment.

An example report from last year's assignment can be found here: `http://sam-dolan.staff.shef.ac.uk/2016/mas212/docs/assign2_report.pdf`