MAS212 Assignment #2:
The dynamics of two-species models

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In this assignment you will investigate the dynamics of biological models, starting with the Lotka-Volterra predator-prey equations,

$$\begin{aligned} \dot{x} &= \quad ax - bxy, \\ \dot{y} &= -cy + dxy. \end{aligned} \tag{1}$$

Here $x = x(t)$ and $y = y(t)$ are functions of time t and $\{a, b, c, d\}$ are positive, real parameters. A dot is used to denote a derivative with respect to time: $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, etc.

The Submission: A completed assignment will comprise a `.pdf` of your report and a single code file (`.py` or `.ipynb`). Your report should comprise four sections corresponding to the four parts below. It should also include a conclusion paragraph, and a list of references. As well as addressing the parts of the brief, your report should also be a coherent document in its own right, which could be read by someone who has not seen this brief. The report should be no more that **seven sides** including figures. If necessary, additional material can be included in an appendix. The report should be accompanied by *one* Python script or Jupyter notebook. Note that the report will carry *much* more credit ($\sim$ 80%) than the code. Please use LaTeX to write your report (or discuss with the lecturer if not possible).

The Deadline: The deadline for submission is found on the course website. Files should be submitted at `http://somas-uploads.shef.ac.uk/mas212`.

Part 1: Introduction

This section should consist of text and mathematics. No code is required for this section.

(a) Motivation. Write a paragraph to introduce and motivate the predator-prey equations (1). Briefly, describe their historical origin. Give a 'biological interpretation' to each parameter $\{a, b, c, d\}$ and the dependent variables $x(t)$ and $y(t)$ (e.g., 'rabbits' and 'foxes'). Describe one way in which the equations **fail** to represent a realistic model. In your introductory text, include three or more citations to appropriate literature (e.g. scientific papers, books or appropriate online resources).

(b) A short proof. Write a paragraph, with equations as necessary, outlining the proof of your choice of **one** of these results:

1. Show that there is a fixed point at $(\bar{x}, \bar{y}) = (c/d, a/b)$ which is a centre; or,

2. By separation of variables, or otherwise, show that C is constant along an orbit, where
$$C \equiv a \ln y - by + c \ln x - dx. \tag{2}$$
or,

3. Show that the temporal averages of $x(t)$ and $y(t)$ along periodic orbits coincide with the fixed-point values $\bar{x}$ and $\bar{y}$.

(c) Rescaled equations. Introduce a new set of variables $\tilde{x} = \alpha x, \tilde{y} = \beta y, \tilde{t} = \gamma t$. With an appropriate choice of constants α, β, γ, show that Eq. (1) may rewritten as the following one-parameter system,

$$\begin{aligned} \frac{d\tilde{x}}{d\tilde{t}} &= \tilde{x} - \tilde{x}\tilde{y}, \\ \frac{d\tilde{y}}{d\tilde{t}} &= -f\tilde{y} + \tilde{x}\tilde{y}, \end{aligned} \tag{3}$$

where f is a real positive parameter which you should express in terms of a, b, c, d.

In the next sections we shall use the variables of part (c), but omit the tilde.

Part 2: The Lotka-Volterra model

In this section, investigate Eqs. (3) by using Python code to compute and plot numerical solutions.

(a) Population dynamics. Use `scipy.integrate.odeint()` to find numerical solutions of Eq. (3), and create the following plots to be included in your report:

(i) a time-domain plot of (t, x) and (t, y) for $t \in [0, 20]$ with initial conditions $x = 1$, $y = 1$, and $f = 2$.

(ii) a phase plot of (x, y) for a range of initial conditions and your choice of f.

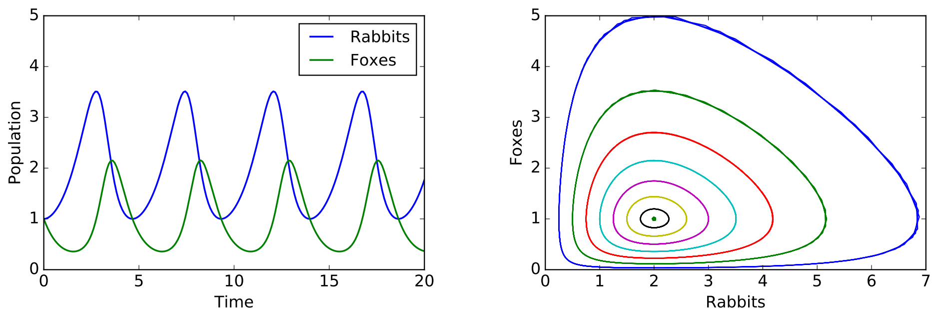

Figure 1: Example plots. *Left:* a time-domain plot, showing predator and prey populations as a function of rescaled time. *Right:* a phase portrait of predators versus prey. The closed loops indicate that the system is periodic. The fixed point (a centre) is at $(f, 1)$, with $f = 2$ in the plots above.

Label the plots, and add a caption. In the report, describe how the plots illustrate typical features of solutions of Eq. (3). Your plots may look similar to those in Fig. 1.

(b) Numerical error. Verify that C is (approximately) conserved along an orbit [see Eq. (2)], by plotting C as a function of time over the interval $t \in [0, 100]$ for an orbit of your choice. Observe that C is not exactly conserved, due to numerical error. Now adjust the error control (the 'tolerance') of `odeint`, using its optional arguments. Add a second line to the plot, showing C for a more accurate numerical solution.

Part 3: Refining the model

Next we consider a modified set of equations,

$$\begin{aligned} \dot{x} &= a(t)x(1 - gx) - \frac{xy}{1 + hx}, \\ \dot{y} &= -fy \quad + \frac{xy}{1 + hx}, \end{aligned} \tag{4}$$

where $\{f, g, h\}$ are constant parameters, and $a(t) = 1$ in this section.

(a) Attractive fixed point. Verify that, for $h = 0$ and $0 < fg < 1$, the system has an attractive fixed point at $(f, 1 - fg)$. Plot a phase portrait to show that, for generic initial conditions, the system evolves towards the limit point. In your report, you may wish to present a labelled plot, or mathematical arguments (or both). What changes when $fg \geq 1$?

(b) Limit cycle. Define 'limit cycle'. Present a phase plot for the case $f = 8/3, g = 3/50, h = 3/20$, showing evidence for a *stable* limit cycle. Try choosing an initial condition close to the fixed point, and comment on what you find. Describe this in your report.

(c) When do limit cycles exist? A mathematician claims that a unique stable limit cycle exists if

$$h > g\frac{1 + fh}{1 - fh}.$$

Investigate this claim. Does it appear to be plausible? Present some evidence, and discuss in your report.

Part 4: A two-species model with harvesting

(More challenging / open-ended)

Now suppose that the prey species is 'harvested' by an outside agent, such as humans. We may (crudely) model this effect by introducing a harmonic term in the prey's reproduction efficiency, that is, by setting $a(t) = 1 + A \sin \omega t$ in Eq. (4), where the amplitude A and frequency ω are two real parameters. Note that the new two-dimensional system is not autonomous, and therefore a 2D phase portrait is less instructive (as 2D trajectories may now cross over). Instead, the system may be written as an **autonomous 3-dimensional** ODE system by introducing a new dependent variable z through $\dot{z} = \omega$. Three-dimensional systems can exhibit rich dynamics (cf. the Lorenz attractor), though the 3D phase space is harder to plot/visualize.

In this section, you should investigate the properties of the system with harvesting, addressing a key question: is the long-term behaviour of the predator-prey populations **periodic** in t? (N.B. the answer may depend on the choice of parameters). Describe your investigation and key findings in no more than 2–3 sides, including any plots.

Pay particular attention to the late-time, long-term behaviour of the system, which is more interesting than the 'transient' response at early times to particular initial conditions. Examine the limit cycle (where it exists). Is the limit cycle periodic? If so, what is its frequency? How does the limit cycle frequency compare to the harvesting frequency ω? How does the behaviour of the system change with A and ω? Are there abrupt transitions?

To get started, you might like to examine the following parameter choices $f = 8/3$, $g = 3/50$, $h = 3/20$ and (i) $\omega = 1$ and $A = 0.61, 0.62, 0.63, 0.64$, (ii) $\omega = 1.5$, $A = 0.05$ and $A = 0.06$, (iii) $\omega = 1.5$ and $A = 0.739$ and $A = 0.7395$.

You may wish to employ more advanced visualisation methods, such as 3D plots, animations, Poincaré sections, and/or bifurcation diagrams. You may want to consider whether concepts such as ‘frequency-locking’, ‘period-doubling’, ‘deterministic chaos’, and/or ‘strange attractor’ are relevant to this system (if so, describe how in your report). Please include a short working definition for any technical terms that you employ.

You may also wish to formulate and investigate your own questions (e.g., “when does a species become extinct in this model?”)

Conclusion and reference list

Add a short conclusion to your report. This should be four sentences or so, summarising the most important findings. Include a list of references.

Guidance:

This assignment will count for $\sim 30\%$ of your module mark, and thus it should be a substantial piece of work. For this assignment, **fair means** include: asking the lecturer for advice; reading online materials; using and adapting short code snippets from lectures; etc. Please avoid verbatim copying, even in the introduction, and please cite all sources used. **Unfair means** include (but are not limited to): sharing or distributing files; copying-and-pasting from work that is not your own; posting your work online; passing off other’s work as your own, etc.

Note that all submissions will be checked for excessive similarities. Where there is circumstantial evidence of unfair means, I reserve the right to award zero marks for this assignment.

An example report from last year’s assignment can be found here: `http://sam-dolan.staff.shef.ac.uk/2015/mas212/docs/assign2_report.pdf`