MAS212 Warm-up exercise for Assignment #1

Dr. Sam Dolan (s.dolan@sheffield.ac.uk)

1. Your friend, a biologist, is looking for help. "For my experiment, I *really* need to find the value of this function $f(x)$ at $x = 10$, but I don't know where to start! Can you help me?"

$$f(x) = \int_0^\infty \frac{e^{-xt}}{1+t} dt, \qquad x \in \mathbb{R},\ x > 0. \tag{1}$$

– **Q.** Is this integral well-defined? Is $f(x)$ finite? Is $f(x)$ positive or negative? What is the range of the function? Sketch the integrand.

2. A passing physics student loudly exclaims "That's easy! You can use the Taylor series expansion of $f(x)$ which I have cleverly derived!"

$$S(x) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots \tag{2}$$

– **Q.** Is this really a Taylor series? Does the series converge? If so, what is its radius of convergence?

3. "Hold on a minute," you interject. "Are you sure this is right?" The physics student, who is often in error but never in doubt, explains his argument:

- First, expand $\frac{1}{1+t}$ as a binomial series: $(1 + t)^{-1} = 1 - t + t^2 - t^3 + \ldots$
- Now consider each term in the integrand of Eq. (1) separately, and use integration by parts to see that

$$\int_0^\infty t^n e^{-xt} dt = \frac{n!}{x^n}.$$

- Add up the terms, and voilà!

– **Q.** Can you find a flaw in your cunning colleague's reasoning?

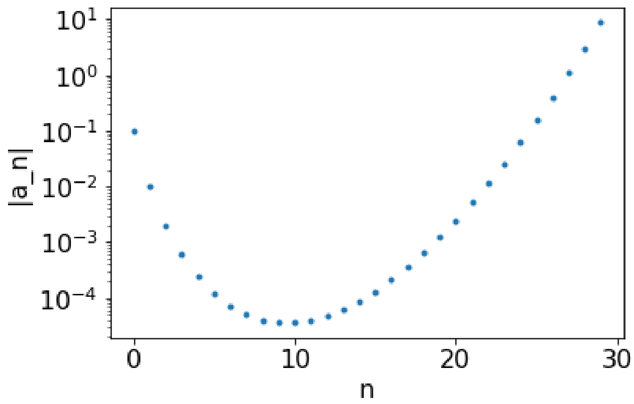

Figure 1: The absolute value of the terms in the series (2) for $x = 10$, shown on a logarithmic scale.

4. A passing Professor mumbles "Ah, an asymptotic series, very interesting stuff ..." and wanders off. The physics student cheers up a little at this, and starting claiming once again to have solved the problem.

– **Q.** Is the series meaningless? Or is it possible that this series may still be useful in practice?

5. Later a kindly Ph.D student explains a little more. "The partial sums of this asymptotic series (2) approximate the function $f(x)$ with an error that is numerically smaller than the first neglected term of the series."

– **Q.** Can you prove this? You could start by defining the error at the nth step, $\epsilon_n = f(x) - S_n$, where S_n is the sum of the first n terms of (2).

6. "How accurately do you need to know the answer for $f(10)$?", you ask your friend. "Oh, just the first three decimal places should be fine," comes the reply. "OK, let me figure out how to use that series to find a best estimate for $f(10)$, and I'll give you an upper bound on the error in the estimate."

– **Q.** How would you calculate this using Python?
– **Q.** Can you think of another way to calculate $f(x)$ using Python?