MAS212 Scientific Computing and Simulation

#5: Numerical methods for ODEs: (a) explicit methods

`http://sam-dolan.staff.shef.ac.uk/mas212`

Key resources:

- Lec 5: `http://sam-dolan.staff.shef.ac.uk/mas212/docs/l5.pdf`
- `http://en.wikipedia.org/wiki/Numerical_methods_for_ordinary_differential_equations`

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

1. Difference equations. Consider the difference equation

$$x_{k+1} = x_k + 2h\sqrt{x_k} \qquad (^\dagger)$$

and $t_{k+1} = t_k + h$.

(a) Let $h = 0.1$. Use (†) to find x_k and t_k for $k = 0 \ldots 20$, starting with $t_0 = 0$ and $x_0 = 0.2$. Plot x_k against t_k. Now add lines on the plot for initial conditions $x_0 = 0.4$, $x_0 = 0.6$, $x_0 = 0.1$ and $x_0 = 0$. (At this point you may wish to write a function which takes h and x_0 as parameters, and returns arrays of t and x values).

(b) Eq. (†) is obtained by applying **Euler's method** to a first-order ODE. Deduce the corresponding ODE and solve it analytically by separation of variables. Compare the exact solutions with your numerical solutions (and comment on the $x_0 = 0$ numerical solution.)

(c) The ratio test. For $x_0 = 0.2$, obtain three data sets: (i) $h = 0.1$ and $k = 0 \ldots 20$; (ii) $h = 0.05$ and $k = 0 \ldots 40$; (iii) $h = 0.025$ and $k = 0 \ldots 80$. Plot the three solutions. Which is most accurate, and why? Now compute the ratio:

$$r_k = \frac{x_k^{(\mathrm{i})} - x_{2k}^{(\mathrm{ii})}}{x_{2k}^{(\mathrm{ii})} - x_{4k}^{(\mathrm{iii})}}, \qquad k > 0.$$

Plot r_k against $t_k^{(\mathrm{i})}$. What do you observe? (Now try halving all values of h, and plot again).

(d) Try changing the $\sqrt{x_k}$ term in (†) to something else and repeat step (c). Why is the ratio approximately constant, and why does it take this particular value?

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2. The midpoint method.

Write a function to implement the midpoint method

$$\begin{aligned} x_{k+1/2} &= x_k + \frac{1}{2} h\, f(x_k, t_k), \\ x_{k+1} &= x_k + h\, f(x_{k+1/2}, t_{k+1/2}), \end{aligned}$$

and use it to obtain a numerical solution of the ODE $\frac{dx}{dt} = \sin^3(x)$.

(a) Plot the solution with $h = 0.1$ across the domain $t \in [0, 20]$, i.e. for the sequence of values $t_k = kh$.

(b) Apply the ratio test, as in 1(c). What do you find? What happens when x passes through zero?

(c) Now try solving $\frac{dx}{dt} = \sin^n(x)$ where n is large. What do you observe? How small does h have to be to resolve the key features?

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3. A second-order equation. The van der Pol equation may be written in first-order form as

$$\dot{x} = y, \quad \dot{y} = \mu(1 - x^2)y - x.$$

Show that application of Euler's method to these equations leads to the difference equations

$$\begin{aligned} x_{k+1} &= x_k + hy_k \\ y_{k+1} &= y_k + h\left(\mu(1 - (x_k)^2)y_k - x_k\right). \end{aligned}$$

where $t_{k+1} = t_k + h$.

(a) Write a function to solve the van der Pol equation with Euler's method. For $\mu = 1$ and a variety of initial conditions, plot of x against y (i.e. make a phase plot). Can you resolve the limit cycles using Euler's method, or is a more sophisticated method required?

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4. The third-order Runge-Kutta method.

Suppose we have an ODE of the form $\frac{dx}{dt} = f(x, t)$. Kutta's third-order method is:

$$x_{n+1} = x_n + \tfrac{1}{3}h\left(k_1 + k_2 + k_3\right)$$

where

$$\begin{aligned} k_1 &= f\left(x_n,\, t_n\right) \\ k_2 &= f\left(x_n + \tfrac{1}{4}hk_1,\, t_n + \tfrac{1}{4}h\right) \\ k_3 &= f\left(x_n - \tfrac{2}{3}hk_1 + hk_2,\, t_n + \tfrac{1}{3}h\right) \end{aligned}$$

Implement Kutta's method, and use it to solve the equation

$$\frac{dx}{dt} = 2\sqrt{x} + \sin(t).$$

Apply the ratio test to show that the method is third-order-accurate.

Answers: 1(b) $\frac{dx}{dt} = 2\sqrt{x}$, with solution $x(t) = (t + \sqrt{x_0})^2$, where x_0 is a constant. NB The numerical solution for $x_0 = 0$ is not what we might expect – why? 1(c) The ratio is ~ 2. As $h \to 0$, the ratio should approach 2. 1(d) The value of the ratio is a property of the method, rather than the particular ODE chosen. The ratio is $r \sim 2^b$ where b is the order of accuracy of the method (meaning that the global truncation error scales with h^b). Hence $r \sim 2$ implies that Euler's method is only first-order accurate.